Exact solutions of nonlinear equations

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Based the homogeneous balance method, a general method is suggested to obtain several kinds of exact solutions for some kinds of nonlinear equations. The validity and reliability of the method are tested by applying it to the Bousseneq equation.

Keywords: Nonlinear equation, Bousseneq equation, Homogeneous balance method, Exact solution.

Nonlinear PDE(partial differential equations) are widely used to describe complex phenomena in various fields of science, especially in physical science. A vast variety of methods including Bäcklund transform[1], Inverse Scattering theory[2] and Hirota's bilinear methods[3] has been developed to obtain analytic solutions of nonlinear PDE. But in some cases[4,5], they cannot do very well. So the simple and direct methods to find analytic solutions of PDE have drawn a lot of interest, for example, the truncated Painlevė expansion[6,7], the hyperbolic tangent function-series method[8,9] and homogeneous balance method[10,11], *etc.* However, only solitary wave solutions are found in most of those methods.

Recently, Wang[10,11] showed the homogeneous balance method is powerful for finding analytic solitary wave solutions of PDE. The essence of the homogeneous balance method can be presented in this way, the nonlinear PDE is

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots\cdots) = 0. \tag{1}$$

If the solution of equation(1) has the form

$$u = \sum_{i=0}^{N} a_i f^{(i)}(\omega(x,t)), \tag{2}$$

where i is an integer, a_i is coefficient, $f(\omega(x,t))$ is a function of $\omega(x,t)$ only, $\omega(x,t)$ is a function of x and t, (i) represents derivative index. The nonlinear terms and the highest order partial derivative terms ought to be partially balanced according to the assumption of homogeneous balance, so N is obtained. Assuming the form of $f(\omega(x,t))$ is

$$f = b\ln(1 + e^{\alpha x+\beta t+\gamma}). \tag{3}$$

Putting (2) and (3) into (1), then deciding coefficients $a_i, b, \alpha, \beta, \gamma$, thus the solitary ware solutions of a given nonlinear PDE are obtained.

This paper noted that the original equation became a set of equations by using the homogeneous balance method, and we attempt to get other kinds of exact solutions except solitary wave solution by solving the set of equations. When N is determined by balancing the nonlinear terms and the highest order partial derivative terms, the form of $f(\omega(x,t))$ and the relation of $f^{(i)} \cdot f^{(j)}$ and $f^{(i+j)}$ can be derived. Then substitute the form of $f(\omega(x,t))$ and the relation of $f^{(i)} \cdot f^{(j)}$ and $f^{(i+j)}$ into equation (1), we get

$$F(f', f'', \ldots \omega_x, \omega_{xx}, \ldots \omega_t, \omega_{tt}, \ldots \omega_{xt}, \ldots) = 0, \tag{4}$$

where F is a function of $f', f'', \ldots \omega_x, \omega_{xx}, \ldots \omega_t, \omega_{tt}, \ldots \omega_{xt}, \ldots$. Obviously, F is a linear polynomial of $f', f'', \ldots$. Setting the coefficients of $f', f'', \ldots$ to zero yields a set of partial differential equations for $\omega(x,t)$. Take note of the term including f' in formula (4), its expression must be $f^{'}\omega_{x_1,x_2,\ldots x_p,t_1\ldots t_q}$. So the coefficient of $f^{'}$ in formula (4) is a linear polynomial $\sum_{i=1}^{k} a_i\omega_{x_1,x_2,\ldots x_{pi},t_1\ldots t_{qi}}$, then the set of the partial differential equations has an important characteristic, namely the equation from the coefficients of f' is a linear partial differential equations of $\omega(x,t)$. Other equations can be regarded as limited conditions to the linear equation, thus solving the set of equations becomes solving the linear PDE with some limited conditions.

In this paper we choose Bousseneq equation[12]

$$u_{tt} - u_{xx} - a(u^2)_{xx} + bu_{xxxx} = 0, \tag{5}$$

to illustrate our method, where a, b are real constants. The bousseneq equation(the wave motion propagate in both directions) describes one-dimensional weakly nonlinear dispersive water waves and can be derived from Toda Lattice,

a suitable approximation enables the KdV equation(the wave motion is restricted to be one direction) to be derived from this equation.

Supposing the solution is of the form

$$u = \sum_{i=0}^{N} f^{(i)}(\omega(x,t)), \tag{6}$$

substituting (6) into (5), and balancing the nonlinear term $a(u^2)_{xx}$ and the linear term bu_{xxxx}, we can get $N=2$,

$$u = f^{''}\omega_x^2 + f^{'}\omega_{xx}. \tag{7}$$

Substituting (7) into (5), and collecting all homogeneous terms in partial dericatives of $\omega(x,t)$, we have

$$\begin{gathered}[bf^{(6)} - 2af^{'''}f^{'''} - 2af^{''}f^{(4)}]\omega_x^6 + [15bf^{(5)} - 24af^{''}f^{'''} - 2af^{'}f^{(4)}]\omega_x^4\omega_{xx} + \\ [f^{(4)}\omega_t^2\omega_x^2 - f^{(4)}\omega_x^4 + (45bf^{(4)} - 24af^{''}f^{''} - 12af^{'}f^{'''})\omega_x^2\omega_{xx}^2 + (20bf^{(4)} - \\ 8af^{''}f^{''} - 4af^{'}f^{'''})\omega_x^3\omega_{xxx}] + [(\omega_{tt}\omega_x^2 + \omega_t\omega_x\omega_{xt} + \omega_t^2\omega_{xx} - 6\omega_x^2\omega_{xx})f^{'''} + \\ (15bf^{'''} - 6af^{'}f^{''})\omega_{xx}^3 + (60bf^{'''} - 20af^{'}f^{''})\omega_x\omega_{xx}\omega_{xxx} + (15bf^{'''} - 2af^{'}f^{''})\omega_x^2\omega_{xxxx}] + \\ [(2\omega_{xt}^2 + 2\omega_x\omega_{xtt} + \omega_{tt}\omega_{xx} - 3\omega_{xx}^2 + 2\omega_t\omega_{xxt} - 4\omega_x\omega_{xxx})f^{''} + (10bf^{''} - 2af^{'}f^{'})\omega_{xxx}^2 + \\ (15bf^{''} - 2af^{'}f^{'})\omega_{xx}\omega_{xxxx} + 6bf^{''}\omega_x\omega_{xxxxx}] + (\omega_{xxtt} + b\omega_{xxxxxx} - \omega_{xxxx})f^{'} = 0.\end{gathered} \tag{8}$$

Setting the coefficient of ω_x^6 in (8) to zero yields an ordinary differential equation for f

$$bf^{(6)} - 2af^{'''}f^{'''} - 2af^{''}f^{(4)} = 0. \tag{9}$$

Solving (9) we obtain a solution

$$f = -\frac{6b}{a}\ln\omega, \tag{10}$$

which yields

$$\begin{array}{lll} f^{''}f^{'''} = \frac{b}{2a}f^{(5)}, & f^{'}f^{(4)} = \frac{3b}{2a}f^{(5)}, & f^{''}f^{''} = \frac{b}{a}f^{(4)}, \\ f^{'}f^{'''} = \frac{2b}{a}f^{(4)}, & f^{'}f^{''} = \frac{3b}{a}f^{'''}, & f^{'}f^{'} = \frac{6b}{a}f^{''}. \end{array} \tag{11}$$

Substituting (11) into (8), formula (8) can be simplified to a linear polynomial of $f', f'', \ldots$, then setting the coefficients of $f', f'', \ldots$ to zero yields a set of partial differential equations for $\omega(x,t)$,

$$\omega_t^2 - \omega_x^2 - 3b\omega_{xx}^2 + 4b\omega_x\omega_{xxx} = 0, \tag{12}$$

$$\omega_{tt}\omega_x^2 + 4\omega_t\omega_x\omega_{xt} + \omega_t^2\omega_{xx} - 6\omega_x^2\omega_{xx} - 3b\omega_{xx}^3 + 9b\omega_x^2\omega_{xxxx} = 0, \tag{13}$$

$$2\omega_{xt}^2 + 2\omega_x\omega_{xtt} + \omega_{tt}\omega_{xx} - 3\omega_{xx}^2 + 2\omega_t\omega_{xxt} - 4\omega_x\omega_{xxx} - 2b\omega_{xxx}^2 + 3b\omega_{xx}\omega_{xxxx} + 6b\omega_x\omega_{xxxxx} = 0, \tag{14}$$

$$\omega_{xxtt} + b\omega_{xxxxxx} - \omega_{xxxx} = 0. \tag{15}$$

Where equation (15) from the coefficients of f' is a linear PDE.

First, we discuss the travelling wave solution of equation(15). Let $\xi = x - vt$ and $\omega(x,t) = s(\xi)$, equation (15) becomes an ordinary differential equation,

$$bs^{(6)} + (v^2 - 1)s^{(4)} = 0. \tag{16}$$

The solution of equation (16) is easy gotten and there exist some different cases to discuss further.

Case 1, If $\beta = \sqrt{\frac{1-v^2}{b}} > 0$, the solution of equation(15) is

$$\omega(x,t) = s(\xi) = d_0 + d_1(x - vt) + d_2(x - vt)^2 + d_3(x - vt)^3 + d_4e^{\beta(x-vt)} + d_5e^{-\beta(x-vt)} \tag{17}$$

Substituting formula(17) into limited conditions(12)-(14), a set of algebra equations are gotten, solving the set of equations, we get $d_1 = 0$, $d_2 = 0$, $d_3 = 0$, $d_4 = 0$ or $d_5 = 0$. Thus the solutions of equations(12)-(15) are gotten,

$$\omega(x,t) = d_0 + d_4 e^{\beta(x-vt)}; \tag{18}$$

$$\omega(x,t) = d_0 + d_5 e^{-\beta(x-vt)}. \tag{19}$$

Where d_0, $d_4$or d_5 are arbitrary constants. Substitute the solution (18) or (19) into formula(6), the exact solitary wave solutions of equation(5) are obtained,

$$u(x,t) = -\frac{3(1-v^2)}{2a}\cosh^{-2}(\frac{1}{2}(\beta(x-vt) + \ln d_4 - \ln d_0)); \tag{20}$$

$$u(x,t) = -\frac{3(1-v^2)}{2a}\cosh^{-2}(\frac{1}{2}(\beta(x-vt) - \ln d_5 + \ln d_0)). \tag{21}$$

Especially $v = 0$, the stationary solitary wave solutions of equation(5) are obtained,

$$u(x,t) = -\frac{3(1-v^2)}{2a}\cosh^{-2}(\frac{1}{2}(\beta x + \ln d_4 - \ln d_0)); \tag{22}$$

$$u(x,t) = -\frac{3(1-v^2)}{2a}\cosh^{-2}(\frac{1}{2}(\beta x - \ln d_5 + \ln d_0)). \tag{23}$$

Case 2, If $\beta = \sqrt{\frac{1-v^2}{b}} = 0$, the solution of equation(15) is

$$\omega(x,t) = s(\xi) = d_0 + d_1(x-vt) + d_2(x-vt)^2 + d_3(x-vt)^3 + d_4(x-vt)^4 + d_5(x-vt)^5 \tag{24}$$

Substituting formula(24) into limited conditions(12)-(14), a set of algebra equations are gotten, solving the set of equations, we get $d_2 = 0$, $d_3 = 0$, $d_4 = 0$, $d_5 = 0$. the solution of the set of equations(12)-(15) is

$$\omega(x,t) = d_0 + d_1(x-vt). \tag{25}$$

Where d_0, d_1 are arbitrary constants. Putting (25) into (6), we get the exact traveling solution of equation(5),

$$u(x,t) = \frac{6bd_1^2}{a[d_0 + d_1(x-vt)]^2}. \tag{26}$$

Case 3, If $\beta = \sqrt{\frac{1-v^2}{b}} < 0$, the solution of the set of equations(12)-(15) is $\omega(x,t) = 0$, only the trivial solution of equation(5), $u(x,t) = 0$, can be gotten.

Second, we discuss a new kind solution of equation(15). Now using x as an integrating factor, the equation(15) may be integrated once to yield

$$\omega_{xtt} + b\omega_{xxxxxx} - \omega_{xxx} = p(t), \tag{27}$$

where $p(t)$ is an arbitrary function. It's easy to know that the equation(27) has the solution

$$\omega(x,t) = S(\xi) + q(t), \tag{28}$$

where $S(\xi)$ is the traveling wave solution of equation $\omega_{xtt} + b\omega_{xxxxxx} - \omega_{xxx} = 0$, $q(t)$ satisfies the equation $\frac{d^2}{dt^2}(q(t)) = p(t)$. There exist some different cases to discuss further.

Case 4, If $\beta = \sqrt{\frac{1-v^2}{b}} > 0$, the solution of equation(15) is

$$\omega(x,t) = s(\xi) = d_0 + d_1(x-vt) + d_2(x-vt)^2 + d_3 e^{\beta(x-vt)} + d_4 e^{-\beta(x-vt)} + q(t). \tag{29}$$

Substituting formula(29) into limited conditions(12)-(14), a set of ordinary differential equations are gotten, solving the set of equations, we get

$$
\begin{aligned}
d_2 &= 0, \\
d_3 &= \frac{bd_1^2(-1+4v^2)}{12d_4v^2(-1+v^2)}, \\
q(t) &= -\frac{d_1(-1+v^2)}{v}t.
\end{aligned}
\tag{30}
$$

Thus the solutions of equations(12)-(15) are gotten,

$$
\omega(x,t) = d_0 + d_1(x-vt) + \frac{bd_1^2(-1+4v^2)}{12d_4v^2(-1+v^2)}e^{\beta(x-vt)} + d_4e^{-\beta(x-vt)} - \frac{d_1(-1+v^2)}{v}t, \tag{31}
$$

where d_0, $d_1$or d_4 are arbitrary constants. Substitute the solution (31) into formula(6), the exact solution of equation(5) are obtained,

$$
\begin{aligned}
u(x,t) = &\frac{6b(d_1 - d_4e^{-\beta(x-vt)}\beta + \frac{bd_1^2(-1+4v^2)\beta e^{\beta(x-vt)}}{12d_4v^2(-1+v^2)})^2}{a(d_0 + d_4e^{-\beta(x-vt)} - \frac{d_1(-1+v^2)t}{v} + \frac{bd_1^2(-1+4v^2)e^{\beta(x-vt)}}{12d_4v^2(-1+v^2)} + d_1(x-vt))^2} - \\
&\frac{6b(\frac{d_4e^{-\beta(x-vt)}(1-v^2)}{b} + \frac{d_1^2(1-v^2)(-1+4v^2)e^{\beta(x-vt)}}{12d_4v^2(-1+v^2)})}{a(d_0 + d_4e^{-\beta(x-vt)} - \frac{d_1(-1+v^2)t}{v} + \frac{bd_1^2(-1+4v^2)e^{\beta(x-vt)}}{12d_4v^2(-1+v^2)} + d_1(x-vt))}.
\end{aligned}
\tag{32}
$$

Case 5, If $\beta = \sqrt{\frac{1-v^2}{b}} = 0$, the solution of equation(15) is

$$
\omega(x,t) = s(\xi) = d_0 + d_1(x-vt) + d_2(x-vt)^2 + d_3(x-vt)^3 + d_4(x-vt)^4 + q(t). \tag{33}
$$

Substituting formula(33) into limited conditions(12)-(14), a set of ordinary differential equations are gotten, solving the set of equations, we get $d_2 = 0$, $d_3 = 0$, $d_4 = 0$, $q(t) = 2d_1t$. Thus the solutions of equations(12)-(15) are gotten,

$$
\omega(x,t) = s(\xi) = d_0 + d_1(x-vt) + 2d_1t, \tag{34}
$$

where d_0, $d_1$are arbitrary constants. Substitute the solution (31) into formula(6), the exact solution of equation(5) are obtained,

$$
u(x,t) = \frac{6bd_1^2}{a(d_0 + 2d_1t + d_1(x-vt))^2}. \tag{35}
$$

Case 6, If $\beta = \sqrt{\frac{1-v^2}{b}} < 0$, the solution of equation(5) is trivial also.

In summary, based the homogeneous balance method, we have introduced an improved method: solving the original nonlinear PDE become solving the linear PDE with some limited conditions. Obviously, the improved method can be applied to the equations that the homogeneous balance method can be applied to. For instance, we use it to the Bousseneq equation and derive four kinds of exact solution. In case 1, the solutions are solitary wave solution, using Wang's method can get them also; in case 2 and 5, the solutions are traveling wave solutions; in case 4, the solution is new kind solution to be discussed further; in case 3 and 6, the trivial solution is gotten.

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